A Family of Non-commutative geometries

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It is shown that the non-commutativity in quantum Hall system may get modified. The self-adjoint extension of the corresponding Hamiltonian leads to a family of non-commutative geometry labeled by the self-adjoint extension parameters. We explicitly perform an exact calculation using a singular interaction and show that, when projected to a certain Landau level, the emergent non-commutative geometries of the projected coordinates belong to a one parameter family. There is a possibility of obtaining the filling fraction of fractional quantum Hall effect by suitably choosing the value of the self-adjoint extension parameter.

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Non-commutative geometry has been one of the contemporary issues in recent research fields [1–4] in theoretical physics. The simplest non-commutativity one assumes is of the form

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}\,, \tag{1}$$

where the real constant $\theta_{\mu\nu}$ is the strength of non-commutativity. It is believed that the effect of spacetime non-commutativity would be important in Plank length scale, i.e., $\theta_{\mu\nu} \sim l_P^2$, which is far from present day experiments. However one may look into the low energy sector to find some form of non-commutativity. The realization of non-commutative spatial geometry in quantum Hall effect [5–10] is one of them. The key ingredient in quantum Hall effect is the presence of landau levels. One can show that the 2-dimensional spatial geometry in the Lowest landau level becomes noncommutative. The strength of non-commutativity is in general inversely proportional to the magnetic field B applied in the system, implying

$$[x, y] = i\frac{1}{B}\,. \tag{2}$$

One can constrain the system in lowest landau level by taking the zero mass limit [11, 12], $m \to 0$, of the evolving particle or by making the magnetic field very large [13, 14], $B \to \infty$. In these limit the coordinates of a plane becomes conjugate to each other. In this view interaction and/or boundary conditions does not seem to have any effect on the non-commutative structure. However there is another way to achieve non-commutativity in the Landau system. Instead of taking limit on the mass or the applied magnetic field one can project the coordinates on the lowest or any specific Landau level. Then the projected coordinates x_P, y_P becomes conjugate to each other [15] and in some units satisfy

$$[x_P, y_P] = i\frac{1}{B}\,, \tag{3}$$

This point of view is important for the study of fractional quantum Hall effect. Because, if we take into account the interaction between the electrons then the non-commutative structure will in general change. Suppose the interaction between the electrons are of the form $V_I = \lambda^2\mathcal{F}(x)$ and suppose the Hamiltonian of the Landau problem remains essentially self-adjoint with the introduction of the interaction. Then the commutator of the projected coordinates will get modified as

$$[x_P, y_P] = i\Theta(B, \lambda)\,, \tag{4}$$

where the real number $\Theta(B, \lambda)$ is now the strength of non-commutativity which depends on the coupling constant λ of the interaction and the magnetic field B in a complicated way. The exact form of the non-commutativity have to be determined by explicitly solving the corresponding Schrödinger equation. Of course in the limit of vanishing interaction one may obtain

$$\lim_{\lambda\to 0}\Theta(B, \lambda) = \frac{1}{B}\,, \tag{5}$$

which gives the familiar result (3). The effect of interaction on the non-commutativity may have importance in understanding the fractional quantum Hall effect. This can be understood as described in [10] by calculating the minimal area acquired by a particle in the non-commutative plane. Note that the non-commutativity (3) implies that the minimal area in the projected space to be

$$\Delta A \equiv \Delta x_P \Delta y_P \simeq \frac{1}{B}\,, \tag{6}$$

Then for an area A number of states available to the electrons in a Landau level is $M = \frac{A}{\Delta A} = AB$. The

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filling fraction for a system of N number of electrons is given by

$$\nu = \frac{N}{M} = \frac{1}{B}\frac{N}{A}\,, \tag{7}$$

For integral quantum Hall effect ν is some integer and can be described by noninteracting electrons. However for fractional quantum Hall effect interactions between the electrons are important. One should therefore consider the modified non-commutativity described by (4), which then implies the filling fraction to be

$$\nu = \frac{N}{M} = \Theta(B,\lambda)\frac{N}{A}\,, \tag{8}$$

Note that the above conclusion is based on the assumption that the Hamiltonian is essentially self-adjoint. However in actual physical situation the interaction between the electrons may make the Hamiltonian non-selfadjoint at least for the s-waves which is important for fractional quantum Hall effect. We therefore in this article assumes that the addition of interaction makes the the Hamiltonian

$$H_L = (\mathbf{p} + e\mathbf{A})^2/2m_e\,, \tag{9}$$

of the Landau problem non-selfadjoint but it has one parameter family of self-adjoint extensions. Here $\mathbf{A}$ is the magnetic vector potential corresponding to the constant magnetic field B perpendicular to the plane and m_e is the reduced mass of two electron system. The interaction potential between the electrons we consider is of the form

$$V_I = \frac{\lambda^2}{r^2}\,, \tag{10}$$

This interaction has a similarity with the gauge potential $A_i = -\frac{\lambda^2}{r^2}\epsilon_{ij}x^j$, which corresponds to a singular flux tube situated at the origin of the coordinates. This kind of singular potential is important to explain quantum Hall effects. The eigenvalue equation $H\psi = E\psi$ will govern the shifts of the Landau levels due to the interact potential V_I, where

$$H = H_L + V_I\,. \tag{11}$$

The introduction of the potential V_I changes the short distance behavior of the wave-functions ψ, which is responsible for making the Hamiltonian non-self-adjoint. This can be understood from Weyl's limit point-limit circle (LPLC) theory.

Before we actually study the Landau problem with interaction, we here give a brief discussion of the LPLC method. Elaborate discussion on it can be found in the book of Reed and Simon [16]. For an ordinary second order linear differential equation of the form

$$H_W\psi_W \equiv \left(-\frac{d^2}{dx^2} + V_W\right)\psi_W = E_W\psi_W\,, \tag{12}$$

defined in $C_0^\infty(0,\infty)$, the potential V_W has the following characteristics. It is said to be in the limit circle case at zero or at infinity respectively, if for all E_W all solutions of (12) are square-integrable at zero or at infinity respectively. If V_W is not in the limit circle case at any of the two boundaries then it is in limit point case at that boundary. The Hamiltonian H_W is essentially self-adjoint on $C_0^\infty(0,\infty)$ if and only if V_W is in the limit point case at both ends, zero and infinity. It the potential V_W is in the limit circle case at both ends then the deficiency indices of the Hamiltonian H_W are both same, $n_+ = n_- = 2$. Note that the deficiency indices $n_\pm$ are the number of solutions $\psi_W^\pm$ of the deficiency equations

$$\left(-\frac{d^2}{dx^2} + V_W\right)\psi_W^\pm = \pm\beta\psi_W^\pm\,, \tag{13}$$

where β is any complex numbers, but for calculation purpose we will take $\beta = i$. The Hamiltonian, for which $n_+ = n_- = 2$, is not self-adjoint but admits 4-parameter family of self-adjoint extensions. Another situation is when V_W is in the limit point case at one boundary point but limit circle case at another boundary point, then the deficiency space solutions are both same. But this time they are all one, i.e., $n_+ = n_- = 1$. This time however one can have a one parameter family of self-adjoint extensions. Finally when V_W is in limit point case at both ends, then the deficiency space solutions are all zero, i.e., $n_+ = n_- = 0$ and the Hamiltonian is essentially self-adjoint.

Let us now return to our problem given by the Hamiltonian H. In order to apply the above method we identify the explicit form of the potential V_W to be

$$V_W = \frac{\sigma^2 - \frac{1}{4}}{r^2} + \frac{1}{4}\omega_B^2 r^2\,, \tag{14}$$

in radial coordinates, where $\sigma^2 = l^2 + \lambda^2$ and $\omega_B = B/2$. The corresponding radial wave-function ψ_W is part of the full wave-function $\psi = \frac{1}{\sqrt{r}}\psi_W \exp(il\phi)$.

Note that the short distance behavior is dominated by the inverse square potential while the long distance behavior is dominated by the harmonic potential. The solutions at short distance is of the form

$$\lim_{r\to 0}\psi_W \simeq r^{(1/2\pm\sigma)}\,. \tag{15}$$

Both the above solutions are square integrable at $r \to 0$ if σ lies in the interval $\sigma \in (-1,+1)$, which then makes the potential V_W in limit circle case at zero. Outside the interval the potential is in limit point case. The long distance $r \to \infty$ behavior is however unperturbed by the interacting potential V_I. One of the solutions is square-integrable and behaves as

$$\lim_{r\to\infty}\psi_W \simeq e^{-\frac{1}{4}\omega_B r^2}\,. \tag{16}$$

The other solution is not square-integrable, which is therefore not acceptable. The potential V_W is therefore

in the limit point case at $r \to \infty$. Outside the critical interval $\sigma \in (-1, +1)$ the potential is in the limit point case at both ends and therefore the the Hamiltonian is essentially self-adjoint. However in the critical interval, since one end is in limit circle case and other end is in limit point case, H is not self-adjoint but has a one parameter family of self-adjoint extensions. Note that the deficiency space solutions in this case are

$$\psi_W^{\pm} = r^{(\frac{1}{2}+\sigma)} e^{-\frac{\omega_B}{4} r^2} U(\xi^{\pm}, 1+\sigma, \frac{\omega_B}{2} r^2)\,, \tag{17}$$

where $\xi^{\pm} = \mp\frac{i}{2\omega_B} + \frac{\sigma+1}{2}$ and U is confluent hypergeometric function [17]. Existence of one square integrable solution of both kind gives the same conclusion that Hamiltonian has a one parameter family of self adjoint extensions when the coupling σ is in the critical interval. The method of finding deficiency space solutions to construct self-adjoint extension is known as von Neumann method [16]. Some of the problems specifically inverse square problem, which is relevant in this case, have been performed in [18, 19]. Given the domain D_W of the symmetric operator H_W the self adjoint extensions, characterized by $e^{i\alpha}, \alpha \in [0, 2\pi]$, is represented by the domain

$$D_\alpha \equiv D_W + \psi_W^+ + e^{i\alpha}\psi_W^-\,. \tag{18}$$

The radial solution ψ_W explicitly can be written as

$$\psi_W = C_l r^{(\frac{1}{2}+\sigma)} e^{-\frac{\omega_B}{4} r^2} U(\xi, 1+\sigma, \frac{\omega_B}{2} r^2)\,, \tag{19}$$

where

$$C_l = \sqrt{\sqrt{2\omega_B}\sin(\frac{E+\omega_B l}{2\omega_B})\frac{\Gamma(\xi)\Gamma(1-\xi)}{\Psi(\xi)-\Psi(1-\xi)}}\,, \tag{20}$$

is the normalization constant explicitly depends on the eigenvalue E and $\xi = -\frac{E+\omega_B l}{2\omega_B} + \frac{\sigma+1}{2}$. In order to find out the explicit form of the eigenvalue we need to match the behavior of the solution ψ_W at $r \to 0$

$$\lim_{r\to 0} \psi_W = A r^{(\frac{1}{2}+\sigma)} + B r^{(\frac{1}{2}-\sigma)}\,, \tag{21}$$

where

$$\begin{aligned} A &= \frac{\pi}{\sin\pi(1+\sigma)}\frac{1}{\Gamma(\xi-\sigma)\Gamma(1+\sigma)} \\ B &= \frac{\pi}{\sin\pi(1+\sigma)}\frac{1}{\Gamma(\xi)\Gamma(1-\sigma)} \end{aligned}$$

with D_α. The behavior of any function, belonging to the domain D_α, near singularity $r \to 0$ can be found from the behavior of $\psi_W^+ + e^{i\alpha}\phi_W^-$ at short distance, because near singularity the function belonging to the domain D_W goes to zero. So,we can write

$$\lim_{r\to 0} D_\alpha = \lim_{r\to 0}(\psi_W^+ + e^{i\alpha}\psi_W^-) \tag{22}$$

$$\begin{aligned} \lim_{r\to 0} \psi_W^+ &= M r^{(\frac{1}{2}+\sigma)} + N r^{(\frac{1}{2}-\sigma)} \\ \lim_{r\to 0} \psi_W^- &= M^* r^{(\frac{1}{2}+\sigma)} + N^* r^{(\frac{1}{2}-\sigma)} \end{aligned} \tag{23}$$

where

$$\begin{aligned} M &= \frac{\pi}{\sin\pi(1+\sigma)}\frac{1}{\Gamma(\xi^+-\sigma)\Gamma(1+\sigma)} \\ N &= \frac{\pi}{\sin\pi(1+\sigma)}\frac{1}{\Gamma(\xi^+)\Gamma(1-\sigma)} \end{aligned} \tag{24}$$

and M^* and N^* are complex conjugates. Now

$$\lim_{r\to 0} D_\alpha \simeq (M + e^{i\alpha}M^*) r^{(\frac{1}{2}+\sigma)} + (N + e^{i\alpha}N^*) r^{(\frac{1}{2}-\sigma)}\,, \tag{25}$$

Equating the coefficient of (21) with (25) we get

$$\frac{A}{B} = \frac{M + e^{i\alpha}M^*}{N + e^{i\alpha}N^*} \in \mathbb{R} \tag{26}$$

This is the energy eigenvalue equation, which is now function of self-adjoint parameter α. By setting a specific value of α we can get the energy spectrum of the system. For example, two extremum solutions can be analytically found. When the right hand side is zero,

$$E = \omega_B(2n + 1 - \sqrt{l^2+\lambda^2} - l)\,, n \in \mathbb{N}^+ \tag{27}$$

and when the right hand side is infinity

$$E = \omega_B(2n + 1 + \sqrt{l^2+\lambda^2} - l)\,, n \in \mathbb{N}^+ \tag{28}$$

Besides the spectrum, the existence of one parameter family of self adjoint extensions has far reaching implications. One possible implication which is important in the present context is the effect on the non-commutative spatial geometry and fractional quantum Hall effect.

The non-commutativity of the projected coordinates x_P, y_P described in (4) will now becomes a family of non-commutative geometries defined by

$$[x_P, y_P]_\alpha = i\Theta(B, \lambda; \alpha)\,, \tag{29}$$

which is our main result in this paper. It is possible to explicitly evaluate the non-commutativity parameter $\Theta(B,\lambda;\alpha)$ for our case. To get the projected coordinates one need to construct projection operator. In a specific energy sector n_0 the projection operator is

$$\mathcal{O}_P = \sum_{l=0}^{\infty} |n_0,l\rangle\langle n_0,l| \tag{30}$$

Then the two projected coordinates on a plane is given by

$$\begin{aligned} x_P &= \mathcal{O}_P x \mathcal{O}_P = \sum_{l,l'=0}^{\infty} \langle n_0,l'|x|n_0,l\rangle |n_0,l'\rangle\langle n_0,l| \\ y_P &= \mathcal{O}_P y \mathcal{O}_P = \sum_{l,l'=0}^{\infty} \langle n_0,l'|y|n_0,l\rangle |n_0,l'\rangle\langle n_0,l| \end{aligned} \tag{31}$$

with

$$\begin{aligned} \langle n_0,l'|x|n_0,l\rangle &= \Omega_{l',l}(\delta_{l',l+1}+\delta_{l',l-1}), \\ \langle n_0,l'|y|n_0,l\rangle &= -i\Omega_{l',l}(\delta_{l',l+1}-\delta_{l',l-1}) \end{aligned} \tag{32}$$

$$\Omega_{l',l} = C_{l'}C_l\pi \int dr r \psi^*_{W_{n_0,l'}} \psi_{W_{n_0,l}} \tag{33}$$

The commutator of the relative coordinates then yields

$$[x_P,y_P]_\alpha = \Theta(B,\lambda;\alpha) = 2\sum_{l=0}^{l=\infty} |\Omega_{l,l+1}|^2 [|n_0,l+1\rangle\langle n_0,l+1| - |n_0,l\rangle\langle n_0,l|]\,, \tag{34}$$

where $\Omega_{l,l+1}$ involves the eigenvalues and therefore in general depend on the self-adjoint extension parameter. The explicit form can be found as

$$\begin{aligned} &\Omega_{l,l+1} = C_{l+1}C_l\frac{\pi}{\omega_B} \times \\ &\Bigg[\frac{\Gamma(1+\mu+\tilde{\mu}+\varrho)\Gamma(1-\mu+\tilde{\mu}+\varrho)\Gamma(-2\tilde{\mu})}{\Gamma(\frac{1}{2}-\tilde{\kappa}-\tilde{\mu})\Gamma(\frac{3}{2}-\kappa+\tilde{\mu}+\varrho)} {}_3F_2(1+\mu+\tilde{\mu}+\varrho, 1-\mu+\tilde{\mu}+\varrho, \frac{1}{2}-\tilde{\kappa}+\tilde{\mu}; 1+2\tilde{\mu}, \frac{3}{2}-\kappa+\tilde{\mu}+\varrho; 1) \\ &+\frac{\Gamma(1+\mu-\tilde{\mu}+\varrho)\Gamma(1-\mu-\tilde{\mu}+\varrho)\Gamma(2\tilde{\mu})}{\Gamma(\frac{1}{2}-\tilde{\kappa}+\tilde{\mu})\Gamma(\frac{3}{2}-\kappa-\tilde{\mu}+\varrho)} {}_3F_2(1+\mu-\tilde{\mu}+\varrho, 1-\mu-\tilde{\mu}+\varrho, \frac{1}{2}-\tilde{\kappa}-\tilde{\mu}; 1-2\tilde{\mu}, \frac{3}{2}-\kappa-\tilde{\mu}+\varrho; 1)\Bigg] \end{aligned} \tag{35}$$

where $\kappa = \frac{E+\omega_B l}{2\omega}$,$\mu = \frac{\sqrt{l^2+\lambda^2}}{2}$,$\varrho = \frac{1}{2}$,$\tilde{\kappa} = \frac{E+\omega_B(l+1)}{2\omega_B}$,$\tilde{\mu} = \frac{\sqrt{(l+1)^2+\lambda^2}}{2}$. The explicit form of $\Theta(B,\lambda;\alpha)$ now depends on the self-adjoint extension parameter and the magnetic field for a fixed interaction strength between the electrons. This two degree of freedom can be exploited to explain fractional quantum Hall effect. We use the similar line of arguments as is done for the composite fermions [20, 21]. We assume that the presence of interaction and therefore imposition of suitable boundary condition enforces the system paired electrons to have an integer filling fraction. The filling fraction for the paired system has a filling fraction

$$\nu_\alpha = \frac{N}{M} = \Theta(B,\lambda,\alpha)\frac{N}{A}\,, \tag{36}$$

As assumed ν_α is now some integer, say $\nu_\alpha = p$, which imply that the filling fraction for the quantum Hall system is

$$\nu = \frac{N}{AB} = \frac{p}{B\Theta(B,\lambda,\alpha)}\,. \tag{37}$$

Now at a specific magnetic field B the desired fraction could be obtained by tuning the self-adjoint extension parameter α and thereby making $B\Theta(B,\lambda,\alpha)$ to be the required integer or fraction.

To summarize, we discussed the problem of system of two electrons on a plane subjected to perpendicular magnetic field which is relevant for the study of quantum Hall effect. We introduced inverse square potential and studied its effect on the non-commutative structure of the projected coordinates. As a result of one parameter family of self-adjoint extensions of the Hamiltonian

we obtained a one parameter family of noncommutative spatial geometry. Using the freedom of the self-adjoint extension parameter we may get the filling fraction of the fraction quantum Hall effects.

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[1] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001).
[2] X. Calmet and M. Selvaggi, Phys. Rev. **D74**, 037901 (2006).
[3] P. R. Giri, P. Roy, Eur. Phys. J. **C57**, 835 (2008).
[4] P. R. Giri, P. Roy, Eur. Phys. J. **C60**, 157 (2009).
[5] Y. S. Myung and H. W. Lee, hep-th/9911031.
[6] D. Bigatti and L. Susskind, Phys. Rev. **D62**, 066004 (2000).
[7] L. Susskind, hep-th/0101029.
[8] V. P. Nair and A. P. Polychronakos, Phys. Lett. **B 505**, 267 (2001).
[9] C. Duval and P. Horváthy, Phys. Lett. **B 479**, 284 (2000).
[10] F. G. scholtz, B. Chakraborty, S. Gangopadhyay and J. Govaerts, J. Phys.A **38**, 9849 (2005).
[11] G. V. Dunne, R. Jackiw and C. A. Trugenberger, Phys. Rev. **D 41**, 661 (1990).
[12] G. V. Dunne and R. Jackiw, Nucl. Phys. **B** (Proc. Suppl.) **C33**, 114 (1993).
[13] A. de Veigy and S. Ouvry, Nucl. Phys. **B 388**, 715 (1992).
[14] Y. S. Myung and H. W. Lee, hep-th/9910083.
[15] N. Macris and S. Ouvry, J. Phys. **A35**, 4477 (2002).
[16] M. Reed and B. Simon, *Fourier Analysis, Self-Adjointness* (New York :Academic, 1975).
[17] M. Abromowitz, I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).
[18] P. R. Giri, K. S. Gupta, S. Meljanac and A. Samsarov, Phys. Lett. **A372**, 2967 (2008).
[19] P. R. Giri, Phys. Rev. **A76**, 012114 (2007).
[20] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[21] J. K. Jain, Physica **E 20**, 79 (2003).